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AN APPROXIMATE METHOD TO COMPUTE THE NONLINEAR NORMAL MODES AND BIFURCATION BY THE PRINCIPLE OF LEAST ACTION

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An analytical procedure is presented to find approximately the nonlinear normal modes in conservative two-degree-of-freedom system by using the principle of least action and by assuming that the modal curve is straight. The results are compared with those of numerical experiments by utilizing the 4th order Runge-Kutta method, and it is found that there are good agreements between them. By utilizing this procedure, it is demnostrated to compute the normal modes which are analytically extended from the linearized modes and to find the generically or non-generically bifurcated modes which do not have any counterpart in the linear theory.

Key Words: Nonlinear Normal Mode, Generic Bifurcation, Non-Generic Bifurcation, Modal Curve, Homogeneous System, Principle of Least Action

1. INTRODUCTION

We shall be interested in formulating an approximate procedure to compute the normal modes in nonlinear conservative systems having two degrees of freedom. As usual in treating nonlinear vibrations in terms of normal mode, we shall be concerned with motions of large amplitudes, the total number of modes, and the bifuracation phenomenon.

The existence of normal modes has been extensively investigated by (Pak and Rosenberg, 1966). Yen (1974) demonstrated that the normal modes occur in pairs. Johnson and Rand (1979) showed that the normal modes generically occur in pairs, and the non-generic case corresponds to the bifurcation.

The procedure to compute the normal modes and the bifurcation in general nonlinear systems has not been reported. For a special type of nonlinear systems called a symmetric system, Anand (1972) computed the normal modes and the bifurcation. The method is not applicable to other nonlinear systems. The method is based on the assumption that the modal curves of both similar and nonsimilar normal mode are straight. The modal curves of nonsimilar modes computed by the fourth order Runge-Kutta method are found to be approximately straight (Pak and Park, 1988).

The purpose of this paper is to formulate a procedure to approximately find the normal modes by utilizing the principle of least action under the assumption that the modal curves are straight. The total number of normal modes, which a system can possess, will be determined through this procedure.

Examples are illustrated for systems having cubic nonlinearity. The locations of normal modes are calculated and the global distribution of normal modes due to the increase of total energy of the system is also demonstrated. The results obtained by present procedure are compared with computer solutions.

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2. PRELIMINARIES

2.1 Nonlinear Two-Degree-of-Freedom System

Let us consider the nonlinear conservative two degrees of freedom system. The mathematical model of the system consists of two concentrated masses m_1 and m_2, connected to each other by means of massless coupled spring and to the wall of each side by means of massless anchor spring, as shown in Fig. 1.

The spring forces of this system are characterized by odd order polynomial

$$G_i = k_i \Delta + a_i \Delta^3 (i=1, 2, 3)$$

where Δ is elongation of spring beyond its unstretched length. In this paper, the above nonlinear, conservative, two degrees of freedom system is called the system S. And the system is called symmetric system if

$$m_1 = m_2 = m, \quad a_1 = a_3 = a, \text{ and } k_1 = k_3 = k,$$

otherwise, it is called unsymmetric system.

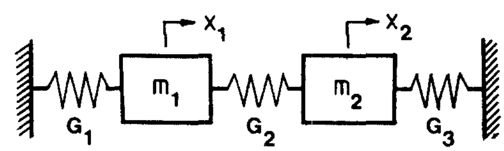

Fig. 1 Non-linear two-degree-of-freedom system

2.2 Definition of Normal Mode

The nonlinear normal mode of two-degree-of-freedom system is defined as follows:

(1) $x_1(t)=x_1(t+\tau)$; every mass does the same periodic motion.

(2) $x_1(t_o)=x_2(t_o)=0$; every mass comes to the equilibrium state simultaneously.

(3) $\dot{x}_1(t_1)=\dot{x}_2(t_1)=0$; every mass has its extreme value at the same time t_1.

(4) $x_2=x_2(x_1)$; x_2 is a single-valued function of x_1 in the closed domain $V(x_1, x_2)=h$ of the x_1x_2-plane.

These normal modes have very inportant meaning in the view that the resonance occurs when the frequency of oscillatory force lies close to the natural frequencies and, in the neighborhood of resonance, linear or not, is subjected to oscillatory forces (Rosenberg, 1966).

The normal modes can be depited in configuration space as shown in Fig. 2. The trajectory is called modal curve. If the modal curve is straight, it is called the similar normal mode, otherwise it is called nonsimilar one (Rosenberg, 1960, 1961).

2.3 Equations of Motion

The kinetic energy T and the potential energy V of the system are in the form

$$T=\frac{1}{2}(m_1\dot{x}_1^2+m_2\dot{x}_2^2) \tag{1}$$

$$V=V_1+V_2 \tag{2}$$

where

$$V_1=\frac{1}{2}\{k_1x_1^2+k_2(x_2-x_1)^2+k_3x_2^2\}$$

$$V_2=\frac{1}{4}\{a_1x_1^4+a_2(x_2-x_1)^4+a_3x_2^4\} \tag{3}$$

The equations of motion can be written in the form :

$$\begin{aligned}m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2+a_1x_1^3-a_2(x_2-x_1)^3=0\\ m_2\ddot{x}_2+(k_2+k_3)x_2-k_2x_1+a_3x_2^3+a_2(x_2-x_1)^3=0\end{aligned} \tag{4}$$

By means of the coordinate transformations

$$x=\sqrt{m_1}x_1$$

and

$$y=\sqrt{m_2}x_2, \tag{5}$$

the kinetic energy T and the potential energy V can be rewritten as.

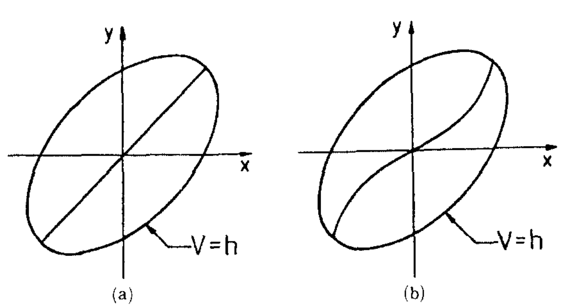

Fig. 2 Types of nonlinear normal mode ;
(a) similar normal mode, (b) nonsimilar normal mode

$$\begin{aligned}T&=\frac{1}{2}(\dot{x}^2+\dot{y}^2)\\ V&=\frac{1}{2}\left\{\frac{k_1}{m_1}x^2+k_2\left(\frac{y}{\sqrt{m_2}}-\frac{x}{\sqrt{m_1}}\right)^2+\frac{k_3}{m_2}y^2\right\}\\ &+\frac{1}{4}\left\{\frac{a_1}{m_1^2}x^4+a_2\left(\frac{y}{\sqrt{m_2}}-\frac{x}{\sqrt{m_1}}\right)^4+\frac{a_3}{m_2^2}y^4\right\}.\end{aligned} \tag{6}$$

Then the equations of motion can be written

$$\begin{aligned}\ddot{x}&=-\frac{\partial}{\partial x}V\left(\frac{x}{\sqrt{m_1}}, \frac{y}{\sqrt{m_2}}\right)\\ \ddot{y}&=-\frac{\partial}{\partial y}V\left(\frac{x}{\sqrt{m_1}}, \frac{y}{\sqrt{m_2}}\right)\end{aligned} \tag{7}$$

Equation (7) are mathematically completely equivalent to equations (4) ; however their physical meanings are quite different. Eqaution (7) may be regarded as the equations of motion of mass points having unit mass that move in the xy-plane.

When moving in the configuration space, the unit mass point traces out a trajectory. Here we can derive the equation of this trajectory. The Eq. (7) can be transformed into the eqaution of trajectory of mass point in the configuration space as follows (Rosenberg, 1966) :

$$2\{h-V(x,y)\}y''+(1+y'^2)(V_y-y'V_x)=0 \tag{8}$$

where h is the total energy of the system.

3. BASIC THEORY

The system considerd in this paper is a holonomic and conservative system. Therefore, the periodic motions of the system can be calculated by applying the Jacobi's principle of least action (Meirovitch, 1970 and Rosenberg, 1977): In holonomic systems possessing an energy integral in which the energy level is fixed in moving from a prescribed initial configuration P_1 to a prescribed terminal configuration P_2, the action integral

$$A=\int_{p1}^{p2}\sqrt{2(h-v)}\,ds, \tag{9}$$

where $ds=\sqrt{dx^2+dy^2}$, along the actual trajectory connecting initial and terminal configurations is stationary relative to all other trajectories connecting the same end configurations, that is,

$$\delta A=0. \tag{10}$$

Thus the principle of least action gives necessary and sufficient conditions for the action A to have a stationary value for the actual motion of the system. Therefore, the normal mode is found by obtaining the point P which make the action A stationary when $P_1=0$ and $P_2=P$. When $ds=\sqrt{\dot{x}^2+\dot{y}^2}dt$, the principle becomes Euler-Lagrange Eq. (7) and when $ds=\sqrt{1+y'^2}\,dx$, the principle results in the equation of trajectory (8).

However, it is very difficult to solve the Eq.(8), since the Eq. (8) is intrinsically nonlinear, even when the springs are linear, and coefficient of y'' can be zero due to the definition of normal modes. But if the normal modes are similar, that is,

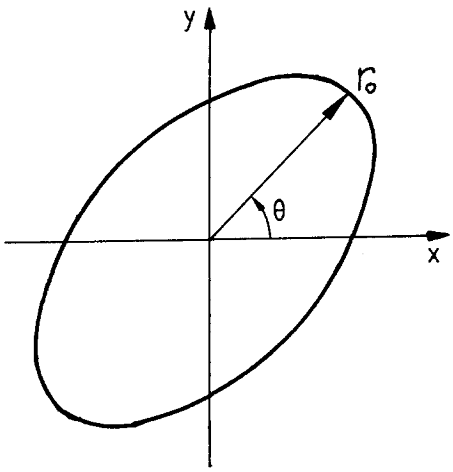

Fig. 3 The coordinate transformation of nonlinear normal mode

$$y' = P(\text{constant}),$$

the calculation becomes easy. The particular nonlinear systems (such as homogeneous and symmetric system) satisfies the above relation. Since the general nonlinear systems do not satisfy above relation, they have the nonsimilai normal modes generally.

On the other hand, although the modal curves of the general nonlinear systems are not straight, it is possible to obtain the approximate nonlinear normal modes by assuming that the modal curve is straight such as Anand's asymmetric modes(Anand, 1972).

Let's compute the normal modes by using the principle of least action, under the assumption that the modal curves of nonlinear normal modes of the system are straight. Let us transform the coordinates $x(t)$ and $y(t)$:

$$x(t) = r(t)\cos\theta$$
$$y(t) = r(t)\sin\theta \tag{11}$$

as shown in Fig. 3 and let us define the parameter r_0 which is the value of $r(t)$ on the curve ($V=h$) given by

$$r_0 = r\left(\frac{\tau}{4}\right)$$

where τ is the period of the normal mode. Then the potential energy of the system can be expressed as

$$V = Dr^2 + Cr^4 \tag{12}$$

where

$$D = \frac{1}{2}\left(\frac{k_1+k_2}{m_1}\cos^2\theta + \frac{k_2+k_3}{m_2}\sin^2\theta - \frac{2k_2}{\sqrt{m_1 m_2}}\sin\theta\cos\theta\right) \tag{13}$$

$$C = \frac{1}{4}\left[\left(\frac{\alpha_1+\alpha_2}{m_1{}^2}\cos^4\theta + \frac{6\alpha_2}{m_1\ m_2}\sin^2\theta\cos^2\theta + \frac{a_2+\alpha_3}{m_2^2}\sin^4\theta\right)\right.$$
$$\left. - \alpha_2\left(\frac{4}{\sqrt{m_1}\sqrt{m_1}^3}\sin^3\theta\cos\theta + \frac{4}{\sqrt{m_1}^3}\sqrt{m_2}\sin\theta\cos^3\theta\right)\right] \tag{14}$$

Since the system S is conservative, the relation

$$V = h \tag{15}$$

is satisfied at the fixed point on the energy curve. Therefore from the Eqs.(12, 15) we can obtain the equation

$$Cr_o^4 + Dr_o^2 - h = 0. \tag{16}$$

Two roots of Eq. (16) with respective to r_o^2 are

$$r_o^2 = -\frac{D}{2C} \pm \left(\frac{D^2}{4C^2} + \frac{h}{C}\right)^{1/2} \tag{17}$$

Now we define the action integral A from the origin 0 to the rest point on the curve $V - \text{h} = 0$:

$$A = \int_o^{ro} \sqrt{2[h - V(r,\ \theta)]}\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr. \tag{18}$$

If we assume that the normal modes are straight, that is, $\frac{d\theta}{dr} = 0$, the Eq.(18) is reduced in the form

$$A = \int_o^{ro} \sqrt{2[h - V(r,\ \theta)]}\, dr.$$

The action integral A is a function of θ only. Therefore, using the principle of least action, we can find the follwing relation

$$\frac{\partial A}{\partial \theta}\delta\theta = 0. \tag{19}$$

Since $\delta\theta$ is an arbitrary value, $\frac{\partial A}{\partial \theta}$ must be equal to zero. Therefore the similar normal modes are satisfied with the relation

$$dA/d\theta = 0. \tag{20}$$

We can compute the nonlinear normal modes of the system from the Eq.(20) as follows.

$$\frac{dA}{d\theta} = -\int_o^{ro} \frac{D'r^2 + C'r^4}{\sqrt{2(h - D_r{}^2 - C_r{}^4)}} d_r = 0 \tag{21}$$

where the primes are denoted the derivative with respect to θ. The integration (Gradshteyn and Ryzhik, 1980) of Eq. (21) leads to the equation

$$3\,D'\left[2\left(\frac{D^2}{4C^2} + \frac{h}{C}\right)^{1/2} E(k) - \left\{\left(\frac{D^2}{4C^2} + \frac{h}{C}\right)^{1/2} + \frac{D}{2C}\right\}K(k)\right]$$
$$+ C'\left[\left(\frac{D^2}{C^2} + \frac{h}{c} + \frac{2D}{C}\left(\frac{D^2}{4C^2} + \frac{h}{C}\right)^{1/2}\right)K(k)\right.$$
$$\left. - 4\frac{D}{C}\left(\frac{D^2}{4C^2} + \frac{h}{C}\right)^{1/2} E(k)\right] = 0 \tag{22}$$

where

$$D' = \left(\frac{k_2+k_3}{m_2} - \frac{k_1+k_2}{m_1}\right)\sin\theta\cos\theta + \frac{k_2}{\sqrt{m_1 m_2}}$$
$$\cos(\sin^2\theta - {}^2\theta) \tag{23}$$

$$C' = -\frac{a^2}{\sqrt{m_1}^3\sqrt{m_2}}\cos^4\theta + \left(\frac{3\alpha_2}{m_1 m_2} - \frac{\alpha_1+\alpha_2}{m_1{}^2}\right)\cos^3\theta\sin\theta$$
$$+ \frac{\alpha_2}{\sqrt{m_1}\sqrt{m_2}^3}\sin^4\theta + 3\alpha_2\left(\frac{1}{\sqrt{m_1}^3\sqrt{m_2}} - \frac{1}{\sqrt{m_1}\sqrt{m_2}^3}\right)$$
$$\sin^2\theta\cos^2\theta + \left(\frac{\alpha_2+\alpha_3}{m_2{}^2} - \frac{3\alpha_2}{m_1 m_2}\right)\cos\theta\sin^3\theta \tag{24}$$

and $K(k)$ and $E(k)$ are the elliptic integrals given by

$$K(k)=\int_0^{\pi/2}\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}d\phi$$
$$E(k)=\int_0^{\pi/2}\sqrt{(1-k^2\sin^2\phi)}\,d\phi \tag{25}$$

where the parameter k^2 is

$$k^2=\frac{\left(\frac{D^2}{4C^2}+\frac{h}{C}\right)-\frac{D}{2C}}{2\left(\frac{D^2}{4C^2}+\frac{h}{C}\right)^{1/2}}$$

We can obtain the angle θ on the configuration space to make the action integral stationary from the Eq(22). Therefore, we can determine the nonlinear normal modes of the form

$$y/x=p=\tan\theta.$$

Now let us determine the nonlinear normal modes of the system due to the variation of the total energy of the system. Firstly, when the total energy of the system is very low, the parameters of the (22) may be simply expressed in the form

$$\left(\frac{D^2}{4C^2}+\frac{h}{C}\right)^{1/2}\doteqdot\left(\frac{C^2}{4C^2}\right)^{1/2}\left(1+\frac{1}{2}\,\frac{h/C}{D^2/4C^2}\right)=\frac{D}{2C}+\frac{h}{D}$$

and

$$k^2\doteqdot\frac{Ch}{D^2},\quad K(k)\doteqdot\frac{\pi}{2}\left(1+\frac{Ch}{4D^2}\right),\quad E(k)\doteqdot\frac{\pi}{2}\left(1-\frac{Ch}{4D^2}\right).$$

Then the Eq.(22) is reduced in the form

$$\frac{dA}{d\theta}=0=6D^2D'-9ChD'+7hDC' \tag{26}$$

Since we assumed that the total energy of the system S is very low, all terms of the Eq(26) except $6D^2D'$ are negligible and D^2 is nonzero. Therefore we can obtain the following fact:

When the total energy h is very low, the normal modes can be obtained from the equation

$$D'=0 \tag{27}$$

which is the equation for determining the mode shapes of the linear systems. Secondly, when the total energy of the system is very high, the first term in the lefthand side of the Eq.(22) can be neglected. This leads to the following relation:

If the total energy of the system is sufficiently high, the nonlinear normal modes can be obtained from

$$C'=0. \tag{28}$$

The solutions satisfied with the Eq.(28) correspond to the normal modes of the associated homogeneous system with degree 4. It is denoted that the homogeneous system with degree $W+1$ is a system which has the potential energy of the form (Rosenberg, 1966)

$$V=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{a_{ij}}{W+1}\left|\frac{x_i}{\sqrt{m_i}}-\frac{x_j}{\sqrt{m_j}}\right|^{W+1}$$

Finally, the nonlinear normal modes of the system S with the general total energy h may be determined from the Eq (22). In this case, numerical computation is effective to solve Eq.(22).

4. EXAMPLES

4.1 Normal Modes of the System with Low Energy

Now we can find the normal modes from the Eq.(27) when the total energy of the system is sufficiently low. The Eq.(27) is rewritten in the form

$$D'=\left(\frac{k_2+k_3}{m_2}-\frac{k_1+k_2}{m_1}\right)\sin\theta\cos\theta+\frac{k_2}{\sqrt{m_1m_2}}(\sin^2\theta-\cos^2\theta)=0 \tag{29}$$

Let $P=\tan\theta$, than Eq.(29) becomes

$$P^2+\frac{\sqrt{m_1}}{\sqrt{m_2}}\left[1+\frac{k_3}{k_2}-\frac{m_2}{m_1}\left(1+\frac{k_1}{k_2}\right)\right]P-1=0. \tag{30}$$

The two roots of the Eq.(30) are

$$P_{1,2}=\frac{1}{2}\left[-\frac{\sqrt{m_1}}{\sqrt{m_2}}\left[1+\frac{k_3}{k_2}-\frac{m_2}{m_1}\left(1+\frac{k_1}{k_2}\right)\right]\pm\sqrt{\frac{m_1}{m_2}\left[1+\frac{k_3}{k_2}-\frac{m_2}{m_1}\left(1+\frac{k_1}{k_2}\right)\right]^2+4}\,\right] \tag{31}$$

It is found that the two roots P_1 and P_2 are satisfied with the orthogonal condition

$$P_1\cdot P_2=-1 \tag{32}$$

Since the square roots in Eq(31) are always positive definite, P_1 and P_2 are always distinct real roots. Therefore we can take the two normal modes $y=P_1x$ and $y\equiv P_2x$. From the orthogonal condition in (32), it is found that one normal mode exists in in-phase and the other mode in out-of-phase.

4.2 Normal Modes of the System with Very High Energy

When the total energy of the system is very high, we can determine the normal modes from the Eq(28). After simple calculations, the Eq. (28) results in the equation

$$P^4+\left(\frac{m_1}{m_2}+\frac{a_3}{a_2}\frac{m_1}{m_2}-3\right)\frac{\sqrt{m_2}}{\sqrt{m_1}}P^3+3\left(\frac{m_2}{m_1}-1\right)P^2+\left(3-\frac{m_2}{m_1}-\frac{a_1}{a_2}\frac{m_2}{m_1}\right)\frac{\sqrt{m_2}}{\sqrt{m_1}}P-\frac{m_2}{m_1}=0 \tag{33}$$

where $P=\tan\theta$. The nonlinear normal modes can be determined by means of solving the Eq.(33).

The possible types of solution of the Eq(33) are as follows; (1) at least two distinct real roots, (2) a simple real root and one triple root, (3) a double root and two distinct real roots, (4) four distinct real roots.

As an example, consider the system which has the very high energy and the coefficients given by

$m_1=m_2=m$. $a_1=a_3=a$, and arbitrary k_1, k_2, k_3 and a_2. Then the Eq. (33) becomes

$$P^4+\left(\frac{a}{a_2}-2\right)P^3+\left(\frac{a}{a_2}-2\right)P-1=0. \tag{34}$$

The Eq.(34) have two or four solutions given by

$$P_{1,2}=\pm 1$$

$$P_{3,4}=\frac{1}{2}\left[\left(2-\frac{a}{\alpha_2}\right)\pm\sqrt{\left(\frac{a}{\alpha_2}-2\right)^2-4}\right].$$

The real solution P_3 and P_4 can exist only when the coefficients have the relation(Ann, 1985)

$$a^2>4a\alpha_2. \tag{35}$$

Let all parameters of the system be positive for convenience, than the solutions of the system correspond to the type (1) when $a<4\alpha_2$, type(2) when $a=4\alpha_2$, and type (4) when $a>4\alpha_2$ in above type of solutions, respectively.

5. BIFURCATION OF NORMAL MODES

5.1 The General Solutions of Equation(22)

In the linear systems, the number of normal modes is the same as the degrees of freedom of the system and the mode shapes are independent on the amplitude of the mode. But in the nonlinear systems, the number of normal modes can exceed that of degree-of-freedom and the mode shapes are dependent on the amplitude or total energy of the system.

Thy system considered in this paper consists of the linear and 4th order nonlinear potential energy. Therefore, when the amplitude is very small, the motion of the system is nearly same as that of the pure linear system. This has been a basic idea that we have analyzed the most physical system with the linearization process. But when the motion has the large amplitude, the motion can not be analyzed by using linear theory. As the amplitude increases, the effects of nonlinear terms on the motion become significant. Finally, when the amplitude of the motion is quite large, the motion approaches the motion of the system with the pure 4th order non-linear potential energy only.

Therefore the solutions of Eq(22) may be changed from the solutions of linearized system as the total energy h of system S increases from very low value. They may be developed from original 2 solutions to 3 or 4 solutions. If the number of roots of the equation $C'=0$ is 2, then the number of normal modes does not change generally, but change their mode shapes continuously as the total energy increases. However, when the number of roots of the equation $C'=0$ is 3 or 4, it is possible that the number of roots of Eq.(22) increases until 3 or 4 and there is obviously a certain energy ho which is an energy of the system when the number of normal modes changes from 2 to 3 or 4. In this case, the bifurcation phenomenon occurs.

5.2 The Bifurcation of Normal Modes of System S

The bifurcation can be classified as the generic bifurcation and the non-generic bifurcation. When the normal modes are bifurcated from the mode extended from a linearized mode, it is called non-generic bifurcation. When the normal modes are bifurcated from a new point, not from an early existing mode, it is called generic bifurcation.

(1) No Bifurcation

When the associated homogeneous system of system S has only two distinct real roots, the linearized modes of the system S vary continuously their mode shapes and approach the modes of homogeneous system with degree 4. This behavior corresponds to the case that the condition (35) is not satisfied.

Table 1 The comparison of the results of several methods to find normal mode for the nonlinear system with the system parameters: ($\alpha_1=\alpha_3=1.6$, $\alpha_2=0.32$, $k_1=k_3=0.2$, $k_2=0.3$, $m_1=m_2=1$)

Energy	Runge-Kutta Method	Present Method	Anand's Method
4.2826	no bifurcation	no bifurcation	−43.4677 −46.5323
4.4334	−43.7646 −46.2354	−44.3168 −45.6832	−41.4118 −48.5882
4.9820	−38.9869 −51.0131	−39.0825 −50.9175	−38.1535 −51.8465
6.8125	−34.1037 −55.8963	−34.1624 −55.8376	−33.6901 −56.3099
15.8790	−28.1807 −61.8193	−28.2169 −61.7831	−28.0020 −61.9980
65.3051	−24.1210 −65.8790	−24.1381 −65.8619	−24.0324 −65.9676
1727.4879	−21.4883 −68.5117	−21.4925 −68.5075	−21.4763 −68.5237

(2) Non-Generic Bifurcation

When the associated homogeneous system of system S has four distinct real roots, the normal modes of a particular nonlinear system such as symmetric systems are non-generically bifurcated from an early existing normal mode at a certain energy ho.

As an example, consider the symmetric system which has the following system parameters

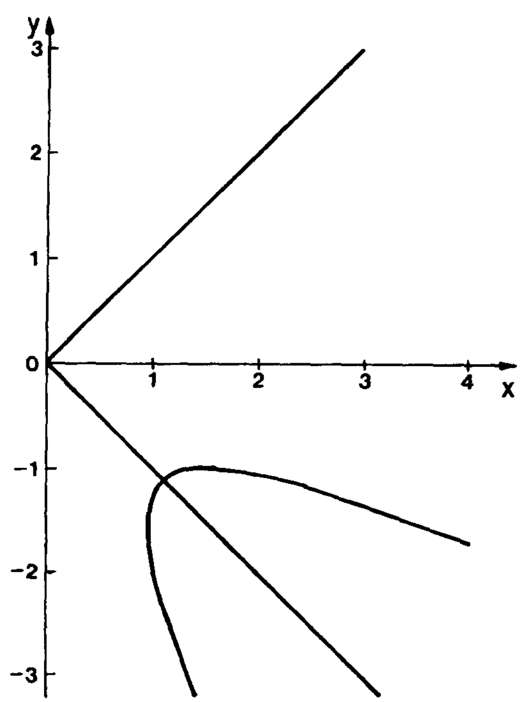

Fig. 4 The trajectories of the rest points of normal modes in the case of non-generic bifurcation.

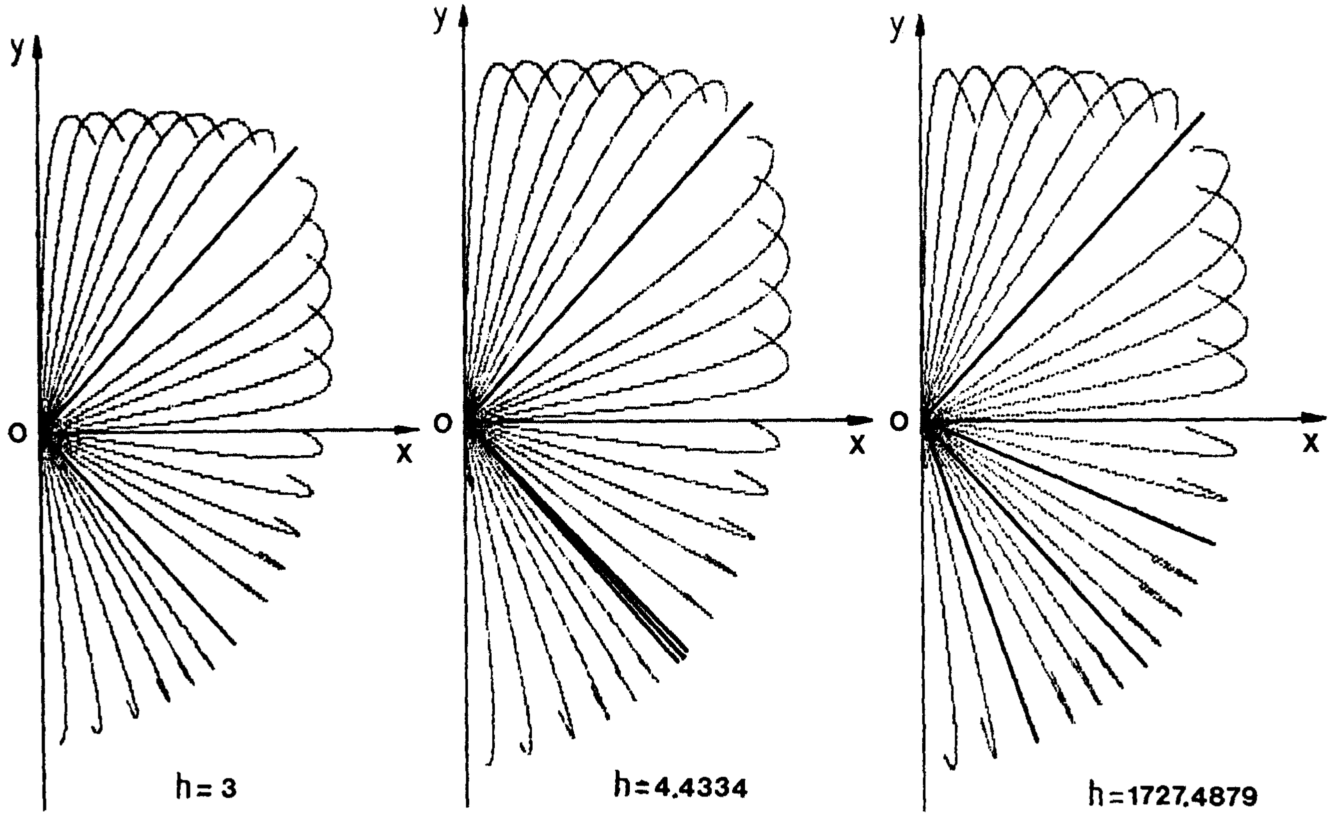

Fig. 5 The modal curves of normal modes (——) and the other trajectories (······) obtained by the 4th order Runge-kutta method in the nongeneric bifurcation case ($m_1=m_2=1,\ k_1=k_3=0.2,\ k_2=0.3,\ a_1=a_3=1.6,\ a_2=0.32$)

$$m_1=m_2=1,\ k_1=k_3=0.2,\ k_2=0.3,\ a_1=a_3=1.6,\ a_2=0.32$$

In this case, the numerical results of this method, Anand's method (1972), and numerical experiment by using the 4th order Runge-Kutta method are compared in the Table 1 where the numbers are the angle of bifurcated modes, measured from the x-axis at the origin. The trajectories of the rest points of normal modes and the modal curves of normal mode obtained by numerical experiment due to the total energy change are plotted in Fig. 4 and Fig 5, respectively. From the above results, we can obtain the following facts:

Table 2 The comparison of the results obtained both by the method proposed in this paper and by numerical method making use of Runge-Kutta method for the system S with the following system parameters: ($a_1=a_3=1.6,\ a_2=0.32,\ k_1=0.4,\ k_2=0.3,\ k_3=0.2,\ m_1=m_2=1$)

Energy	Runge-Kutta Method		Present Method	
4.2826	46.0804	−28.3774	46.0240	−28.4241
6.8125	45.8680	−27.0516	45.8184	−27.0939
15.8790	45.5744	−25.0655	45.5415	−25.0910
There is an energy h_0 such that only a new mode occurs.				
21.3326	45.4941	−24.5119	45.4684	−24.5352
	−55.2347	−56.0526	−55.6031	−55.6154
65.3051	45.2859	−22.9814	45.2695	−22.9965
	−48.0447	−64.4143	−48.0416	−64.3860
1727.4879	45.0542	−21.2957	45.0528	−21.3119
	−45.4510	−68.3250	−45.4496	−68.3196

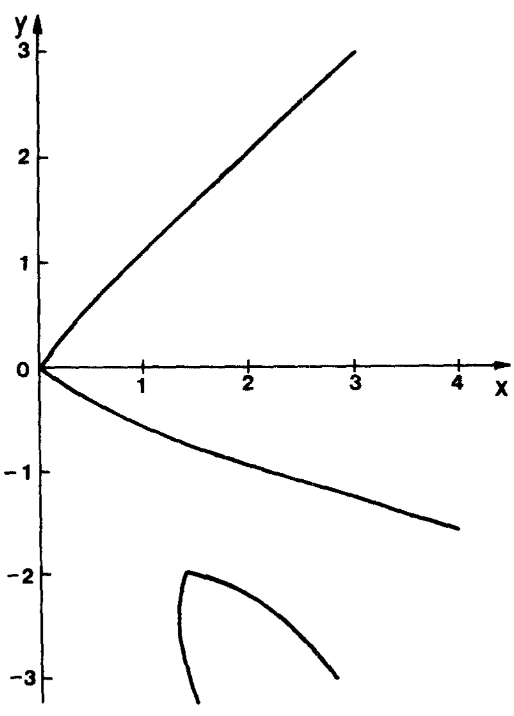

Fig. 6 The trajectories of the rest points of normal modes in the case of generic bifurcation

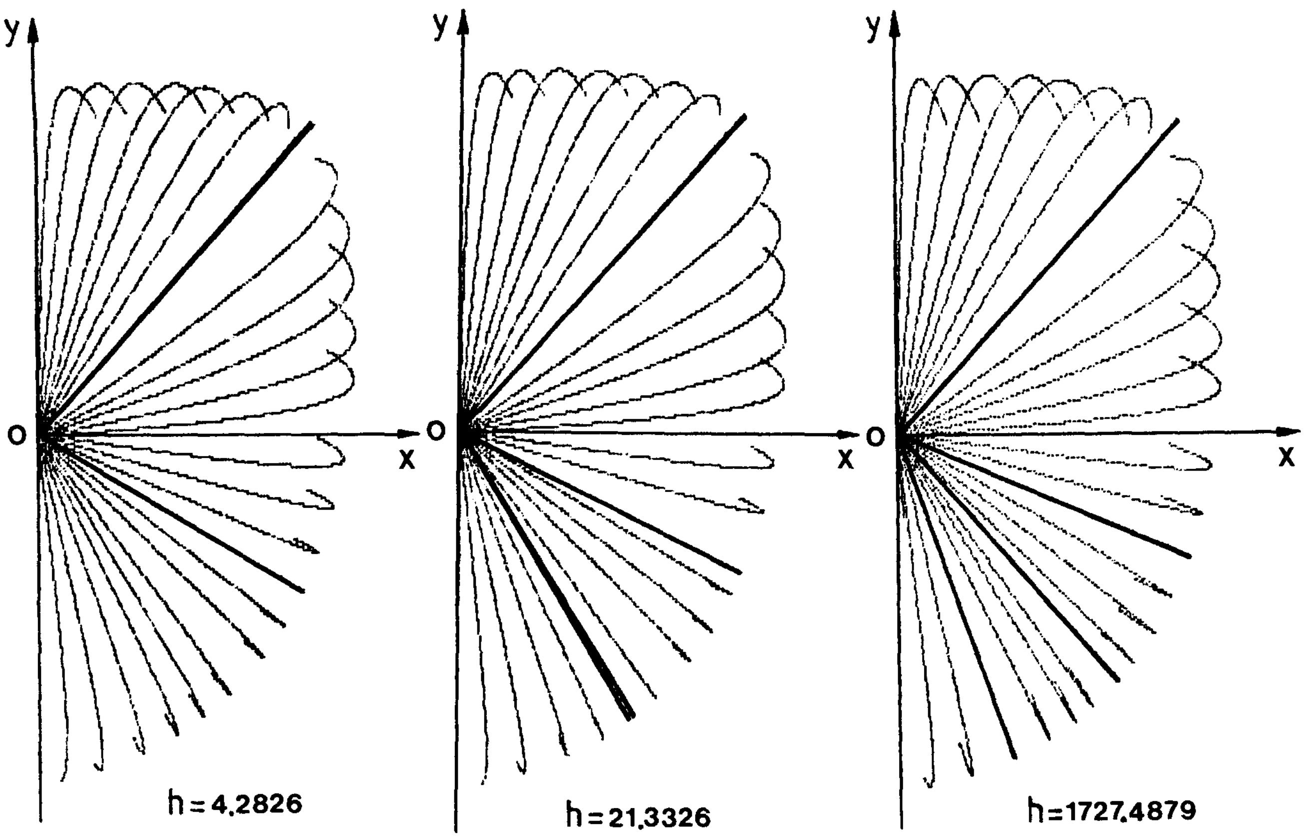

Fig. 7 The modal curves of normal modes (——)and the other trajectories (……) obtained by the 4th order Runge-Kutta method in the generic bifurcation case ($m_1=m_2=1$, $k_1=0.4$, $k_2=0.3$, $k_3=0.2$, $\alpha_1=\alpha_3=1.6$, $\alpha_2=0.32$)

(a) The $y\equiv x$ mode and $y\equiv -x$ mode always exist, if the total energy is high or not.

(b) As the energy increases, the new normal modes occur in pairs from the early existing mode $y\equiv -x$, and symmetrically with respect to the out-of-phase mode $y\equiv -x$, at the energy in the neighborhood of 4.30, as shown in Table 1.

(c) The bifurcated modes change their mode shapes continuously and approach the normal modes of the associated homogeneous system with degree 4, as the total energy increases.

(3) Generic Bifurcation

When the associated homogeneous system with degree 4 of system S has four distinct real roots, the bifurcation of normal modes occurs generally at a new point, not at the mode extended from a linearized mode.

Consider a system, as an example, which has the following parameter values;

$$m_1=m_2=1,\ k_1=0.4,\ k_2=0.3,\ k_3=0.2,\ \alpha_1=\alpha_3=1.6,\ \alpha_2=0.32$$

By the digital computer, the numerical results of present method and the numerical experiment by using the 4th order Runge-Kutta method are tabulated in the Table 2. The trajectories of rest points and the modal curves of normal modes obtained by the numerical experiment are shown in Fig. 6, and Fig. 7, respectively. From these results, we can find the following facts:

(a) Two linearized normal modes exist and vary continuously their mode shapes as the energy h increases.

(b) There is an energy ho at which the normal modes are bifurcated. In this moment, the number of normal modes becomes 3.

(c) As the total energy increases from ho, the bifurcated normal modes vary continuously their mode shapes and approach the corresponding normal mode of homogneous system with degree 4.

(d) Therefore the total number of normal mode becomes 4 when the total energy is sufficiently high.

6. CONCLUSION

An analytical procedure to calculate approximate nonlinear normal modes in nonlinear conservative two-degree-of-freedom system is developed by utilizing the principle of least action. By Applying this method to the nonlinear system with cubic nonlinearity, we can obtain the following facts:

(1) It can be found that results of this approximate method agree very well with those of the computer simulation in spite of the assumption that the normal modes are straight.

(2) There are at least two nonlinear normal modes. The maximum number of nonlinear normal modes can not exceed the highest order of potential energy.

(3) As the total energy increases, the normal modes change their mode shapes continuously from the linearized modes and approach the corresponding normal mode of the associated homogeneous system with the highest degree.

(4) The bifurcation phenomena of normal mode are inves-

tigated. As a result, it can be known that there are two types of bifurcation, that is, generic bifurcation and nongeneric bifurcation. This phenomena have a very important meaning in the view that the resonance occurs in the neighborhood of normal modes.

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